Waves and Optics - PHY204
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5

Electromagnetic Waves
&
Polarization

The one-dimensional wave equation

We can derive the wave equation directly from Maxwell's equations. Here it is in its one-dimensional form for scalar (i.e., non-vector) functions, *f*:

$$\boxed{\frac{\partial^2 f}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2 f}{\partial t^2} = 0}$$

Light waves (actually the electric fields of light waves) will be a solution to this equation. And **v** will be the velocity of light.

The 1D wave equation for light waves

$$\boxed{\frac{\partial^2 E}{\partial x^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0}$$

where E is the light electric field

We'll use cosine- and sine-wave solutions:

$$E(x,t) = A_1 \cos[k(x \pm \mathrm{v}t)] + A_2 \sin[k(x \pm \mathrm{v}t)]$$

$$kx \pm (k\mathrm{v})t$$

$$E(x,t) = A_1 \cos(kx \pm \omega t) + A_2 \sin(kx \pm \omega t)$$

$$\boxed{\frac{\omega}{k} = \mathrm{v} = \frac{1}{\sqrt{\mu\varepsilon}}}$$

The speed of light in vacuum, usually called "c", is 3 x 10^8 m/s.

A simpler equation for a harmonic wave:

$E(x,t) = A\cos[(kx - \omega t) - \theta]$

Use the trigonometric identity:

$\cos(z-y) = \cos(z)\cos(y) + \sin(z)\sin(y)$

where $z = kx - \omega t$ and $y = \theta$ to obtain:

$E(x,t) = A\cos(kx - \omega t)\cos(\theta) + A\sin(kx - \omega t)\sin(\theta)$

which is the same result as before,

$$E(x,t) = A_1\cos(kx \pm \omega t) + A_2\sin(kx \pm \omega t)$$

as long as:

$A\cos(\theta) = A_1$ and $A\sin(\theta) = A_2$

For simplicity, we'll just use the forward-propagating wave.

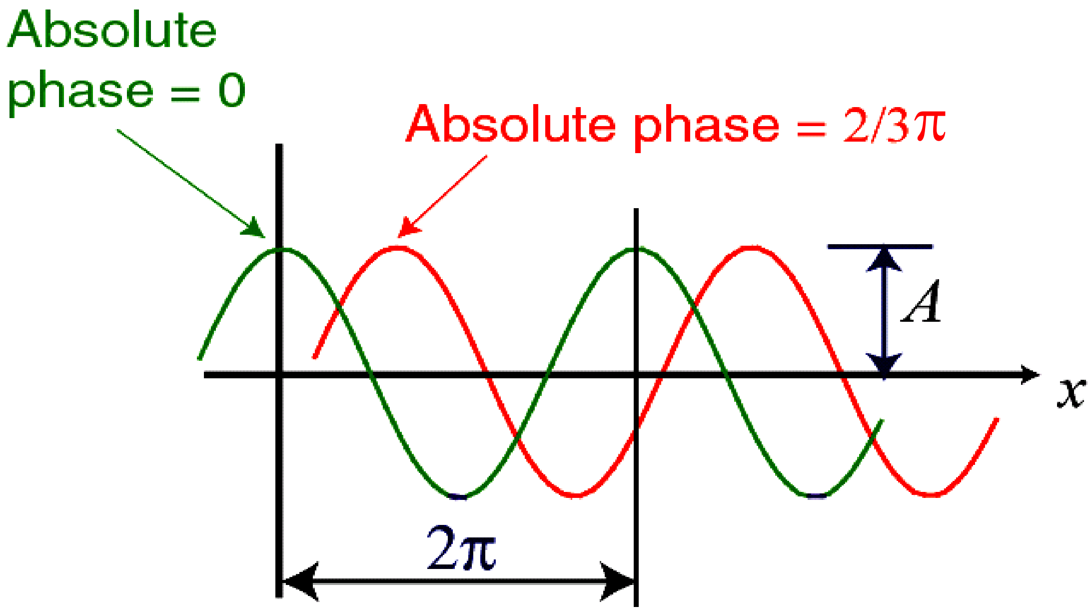
Definitions: Amplitude and Phase

$$\boxed{E(x,t) = A\cos[(k\,x - \omega\,t) - \theta] = A\cos(\varphi)}$$

A = Amplitude

φ = Phase, $\varphi = \varphi(x,y,z,t)$ and is not a constant!

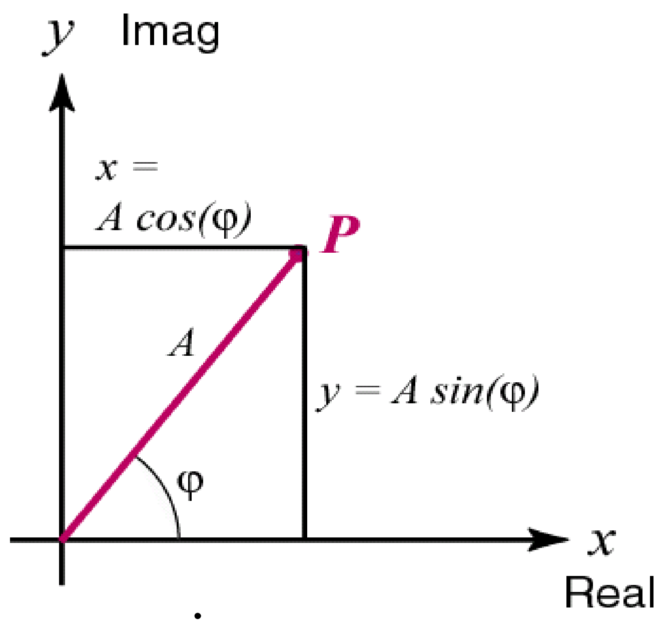
θ = Absolute Phase (or Initial Phase)



Complex numbers

Consider a point, $P = (x,y)$, on a 2D Cartesian grid.

Let the x-coordinate be the real part and the y-coordinate the imaginary part of a complex number.



So, instead of using an ordered pair, (x,y), we write:

$$\boxed{P = x + iy = A\cos(\varphi) + iA\sin(\varphi)}$$

where $i = \sqrt{-1}$

Euler's Formula

$$\boxed{e^{i\varphi} \equiv \exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)}$$

so the point, $P = x + iy = A\cos(\varphi) + iA\sin(\varphi)$

can be written:

$$\boxed{P = Ae^{i\varphi} \equiv A\exp(i\varphi)}$$

where

$$A = \text{Amplitude}$$

$$\varphi = \text{Phase}$$

Proof of Euler's Formula

$$\boxed{e^{i\varphi} \equiv \exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)}$$

Use Taylor Series:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots$$

If we substitute $x = i\varphi$ into $\exp(x)$, then:

$$\exp(i\varphi) = 1 + \frac{i\varphi}{1!} - \frac{\varphi^2}{2!} - \frac{i\varphi^3}{3!} + \frac{\varphi^4}{4!} + \ldots$$

$$= \left[1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \ldots\right] + i\left[\frac{\varphi}{1!} - \frac{\varphi^3}{3!} + \ldots\right]$$

$$= \cos(\varphi) + i\sin(\varphi)$$

Complex number theorems

If $\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$

$$\exp(i\pi) = -1$$

$$\exp(i\pi/2) = i$$

$$\exp(\text{-}i\varphi) = \cos(\varphi) - i\sin(\varphi)$$

$$\cos(\varphi) = \frac{1}{2}\left[\exp(i\varphi) + \exp(-i\varphi)\right]$$

$$\sin(\varphi) = \frac{1}{2i}\left[\exp(i\varphi) - \exp(-i\varphi)\right]$$

$$A_1\exp(i\varphi_1) \times A_2\exp(i\varphi_2) = A_1 A_2 \exp\left[i(\varphi_1 + \varphi_2)\right]$$

$$A_1\exp(i\varphi_1) / A_2\exp(i\varphi_2) = A_1 / A_2 \exp\left[i(\varphi_1 - \varphi_2)\right]$$

More complex number theorems

Any complex number, z, can be written:

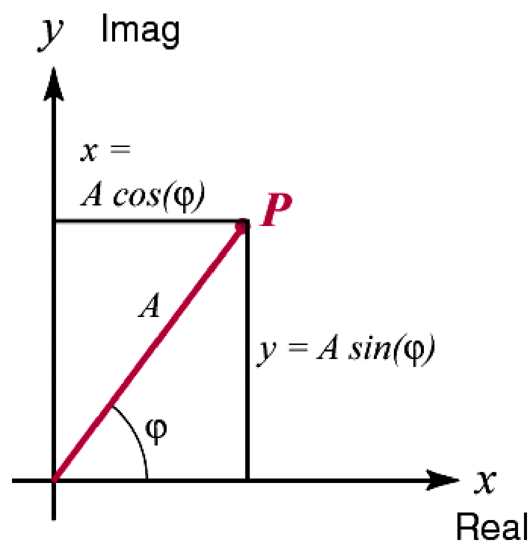
$$z = \mathrm{Re}\{z\} + i\,\mathrm{Im}\{z\}$$

So

$$\mathrm{Re}\{z\} = 1/2\ (z + z^*)$$

and

$$\mathrm{Im}\{z\} = 1/2i\ (z - z^*)$$



where z^* is the complex conjugate of z ($i \rightarrow -i$)

The "magnitude," $|z|$, of a complex number is:

$$|z|^2 = z\,z^* = \mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2$$

To convert z into polar form, $A\ \exp(i\varphi)$:

$$A^2 = \mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2$$

$$\tan(\varphi) = \mathrm{Im}\{z\} / \mathrm{Re}\{z\}$$

Waves using complex numbers

The electric field of a light wave can be written:

$E(x,t) = A\cos(kx - \omega t - \theta)$

Since $\exp(i\varphi) = \cos(\varphi)+i\sin(\varphi)$, $E(x,t)$ can also be written:

$E(x,t) = \mathrm{Re}\,\{A\exp[i(kx - \omega t - \theta)]\}$

or $E(x,t) = ½\,A\{\exp[i(kx - \omega t - \theta)]+\exp[-i(kx - \omega t - \theta)]\}$

We often write these expressions without the ½, Re, or +complex conjugate and we write simply:

$$\boxed{E(x,t) = A\exp[i(kx-\omega t-\theta)]}$$

But $E(x,t)$ **is a vector** $\Rightarrow A$ **is a vector** $=E_0$

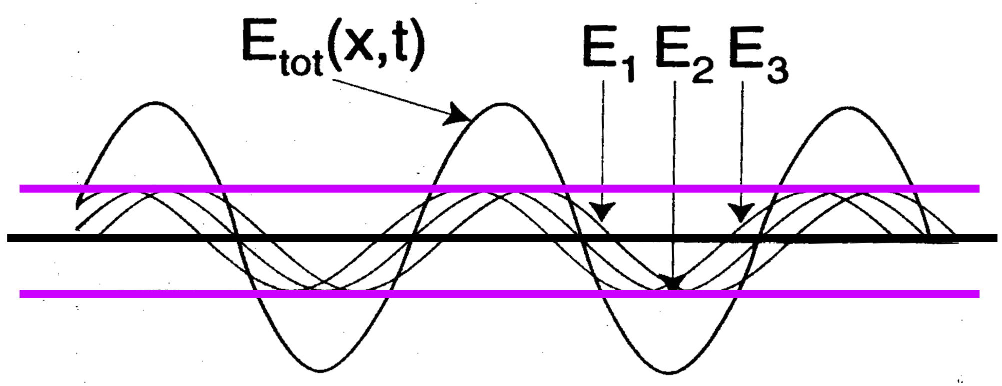
Complex numbers simplify wave optics!

Adding waves of the same frequency, but different initial phase, yields a wave of the same frequency.

This isn't so obvious using trigonometric functions, but it's easy with complex exponentials:

$$\underset{\sim}{E}_{tot}(x,t) = \underset{\sim}{E}_1 \exp i(kx-\omega t) + \underset{\sim}{E}_2 \exp i(kx-\omega t) + \underset{\sim}{E}_3 \exp i(kx-\omega t)$$
$$= (\underset{\sim}{E}_1 + \underset{\sim}{E}_2 + \underset{\sim}{E}_3) \exp i(kx-\omega t)$$

where all initial phases are lumped into E_1, E_2, and E_3.



The 3D wave equation for the electric field

A light wave can propagate in any direction in space. So we must allow the space derivative to be 3D:

$$\boxed{\vec{\nabla}^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0}$$

or $$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

which has the solution: $$\boxed{\underset{\sim}{E}(x, y, z, t) = \underset{\sim}{E}_0 \exp[i(\vec{k}\cdot\vec{r} - \omega t)]}$$

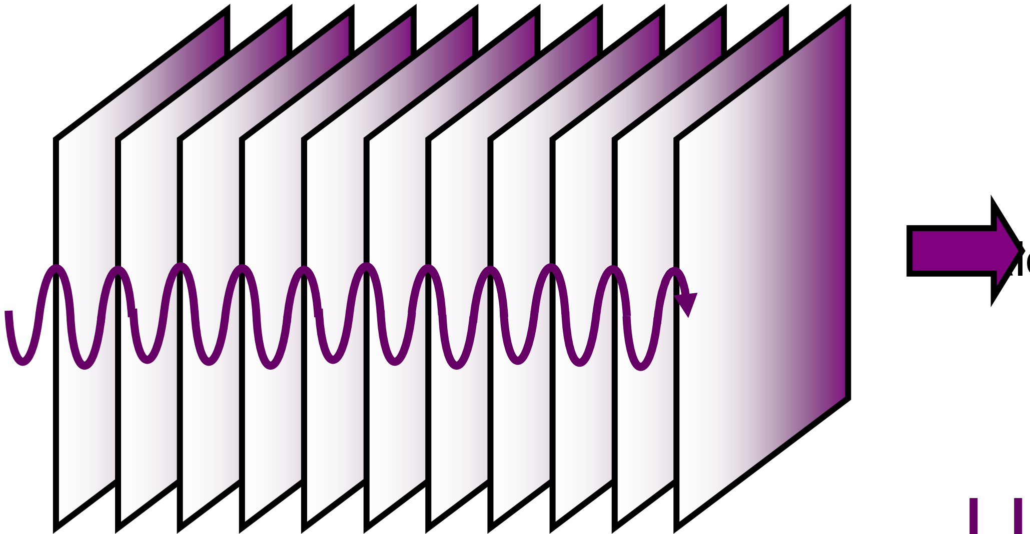
where $\vec{k} \equiv \left(k_x, k_y, k_z\right)$ $\quad \vec{r} \equiv \left(x, y, z\right)$ $\quad k^2 \equiv k_x^2 + k_y^2 + k_z^2$

and $\vec{k}\cdot\vec{r} \equiv k_x\, x + k_y\, y + k_z\, z$

$\underset{\sim}{E}_0 \exp[i(\vec{k}\cdot\vec{r}-\omega t)]$ is called a plane wave.

A plane wave's contours of maximum field, called **wave-fronts** or **phase-fronts**, are planes. They extend over all space.

Wave-fronts are helpful for drawing pictures of interfering waves.



A wave's wave-fronts sweep along at the speed of light.

A plane wave's wave-fronts are equally spaced, a wavelength apart.

They're perpendicular to the propagation direction.

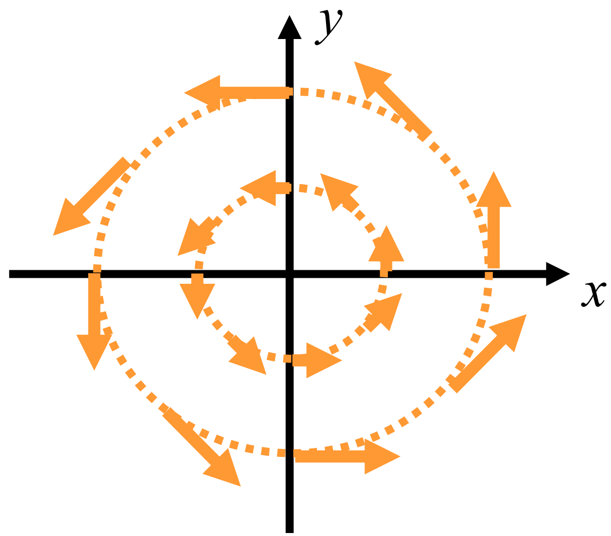
Usually, we just draw lines; it's easier.

Vector fields

Light is a 3D vector field.

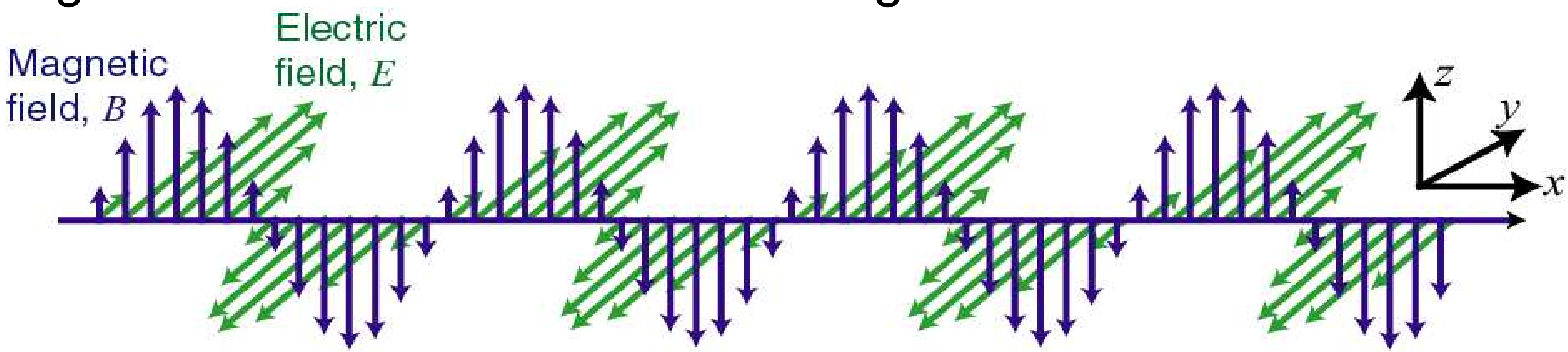
$$\vec{f}(\vec{r})$$

A 3D vector field assigns a 3D vector (i.e., an arrow having both direction and length) to each point in 3D space.



A 2D vector field

A light wave has both electric and magnetic 3D vector fields:



Waves using complex vector amplitudes

We must now allow the complex field ***E*** and its amplitude ***E₀*** to be vectors:

$$\boxed{\underset{\sim}{\vec{E}}\left(\vec{r},t\right)=\underset{\sim}{\vec{E}}_0 \exp\left[i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right]}$$

The complex vector amplitude has six numbers that must be specified to completely determine it!

$$\underset{\sim}{\vec{E}}_0 = (\overbrace{\mathrm{Re}\{E_x\}+i\,\mathrm{Im}\{E_x\}}^{x\text{-component}},\ \overbrace{\mathrm{Re}\{E_y\}+i\,\mathrm{Im}\{E_y\}}^{y\text{-component}},\ \overbrace{\mathrm{Re}\{E_z\}+i\,\mathrm{Im}\{E_z\}}^{z\text{-component}})$$

The equations of optics are Maxwell's equations.

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

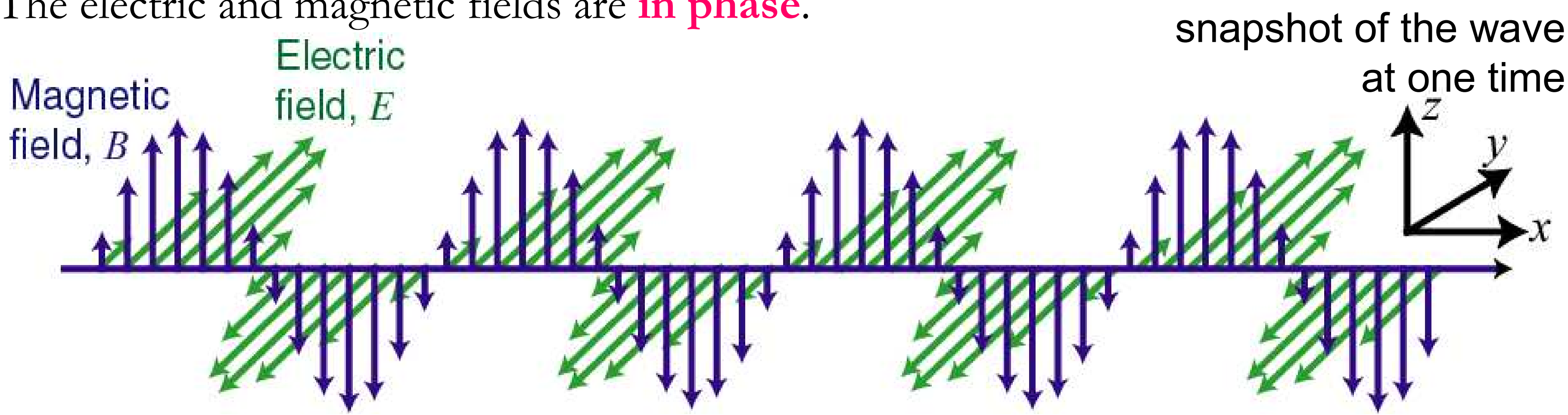
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, ε is the permittivity, and μ is the permeability of the medium.

As written, they assume no charges (or free space).

The magnetic-field strength in a light wave

The electric and magnetic fields are **in phase**.



The **electric field**, the **magnetic field**, and the **k-vector** are all perpendicular:

$$\boxed{\vec{E} \times \vec{B} \propto \vec{k}}$$

$$\boxed{B_z(x,t) = \frac{1}{c} E_y(x,t)}$$

Why we neglect the magnetic field

The force on a charge, q, is:

$$\vec{F} = \underbrace{q\vec{E}}_{\vec{F}_{electrical}} + \underbrace{q\,\vec{\mathrm{v}}\times\vec{B}}_{\vec{F}_{magnetic}}$$

where $\vec{\mathrm{v}}$ is the charge velocity

Taking the ratio of the magnitudes of the two forces:

$$\frac{F_{magnetic}}{F_{electrical}} \le \frac{q\mathrm{v}B}{qE}$$

$$\left|\vec{\mathrm{v}}\times\vec{B}\right| = \mathrm{v}B\sin\theta \le \mathrm{v}B$$

Since $B = E/c$:

$$\frac{F_{magnetic}}{F_{electrical}} \le \frac{\mathrm{v}}{c}$$

So as long as a charge's velocity is much less than the speed of light, we can neglect the light's magnetic force compared to its electric force.

The Irradiance (often called the Intensity)

A light wave's *average* power per unit area is the **irradiance.**

$$\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{T} \int_{t-T/2}^{t+T/2} \vec{S}(\vec{r},t')\, dt'$$

Substituting a light wave into the expression for the Poynting vector, $\vec{S} = c^2\, \varepsilon\, \vec{E} \times \vec{B}$ yields:

real amplitudes

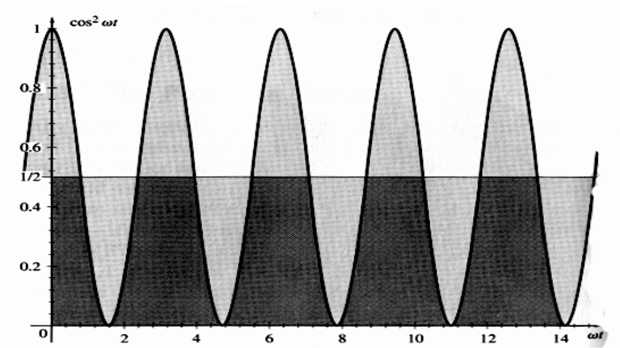
$$\vec{S}(\vec{r},t) \quad = \quad c^2\, \varepsilon\, \vec{E}_0 \times \vec{B}_0 \cos^2(\vec{k}\cdot\vec{r} - \omega t - \theta)$$

The average of $\cos^2$ is 1/2:

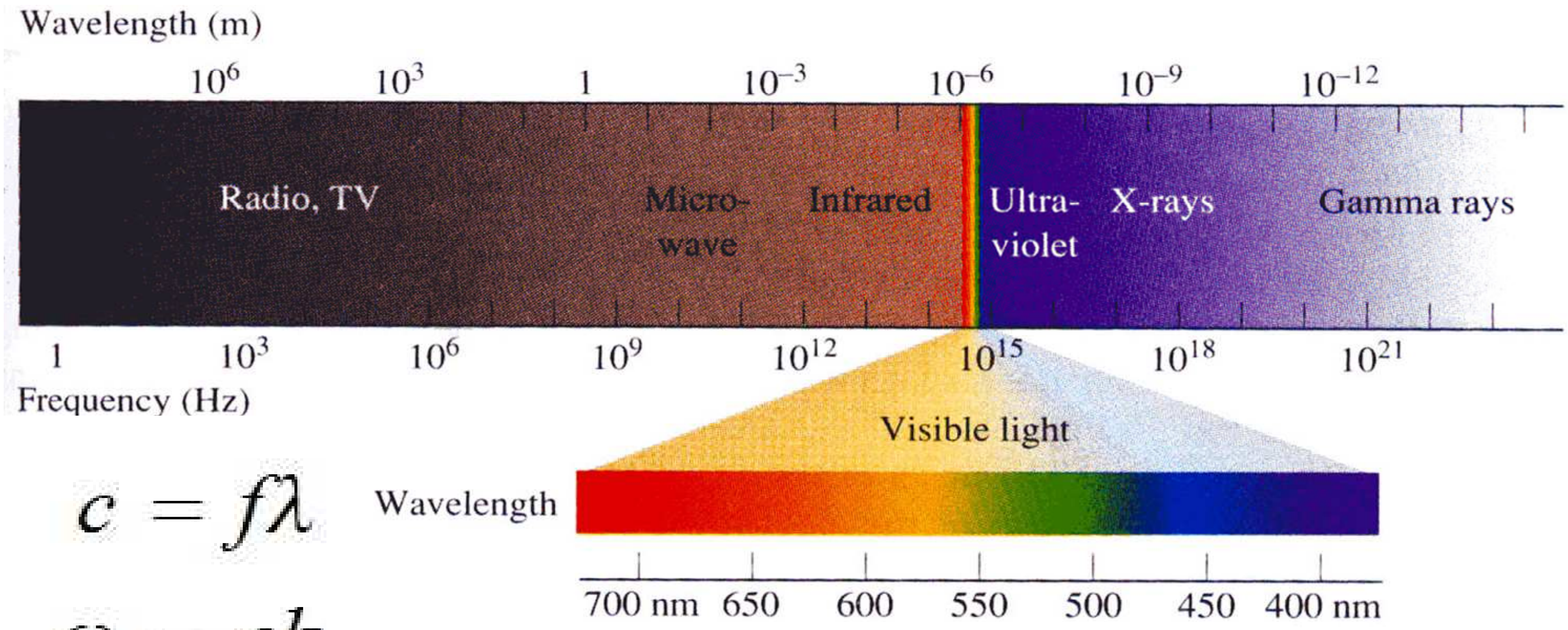
$$\Rightarrow \; I(\vec{r},t) \; = \left| \left\langle \vec{S}(\vec{r},t) \right\rangle \right| =$$

$$= \; c^2\, \varepsilon \left| \vec{E}_0 \times \vec{B}_0 \right| (1/2)$$

$$= \frac{1}{2} c^2 \varepsilon \left| E_0 \right| \frac{\left| E_0 \right|}{c} = \frac{1}{2} c \varepsilon \left| E_0 \right|^2$$



Electromagnetic Spectrum -1



$$c = f\lambda$$

$$\omega = ck$$

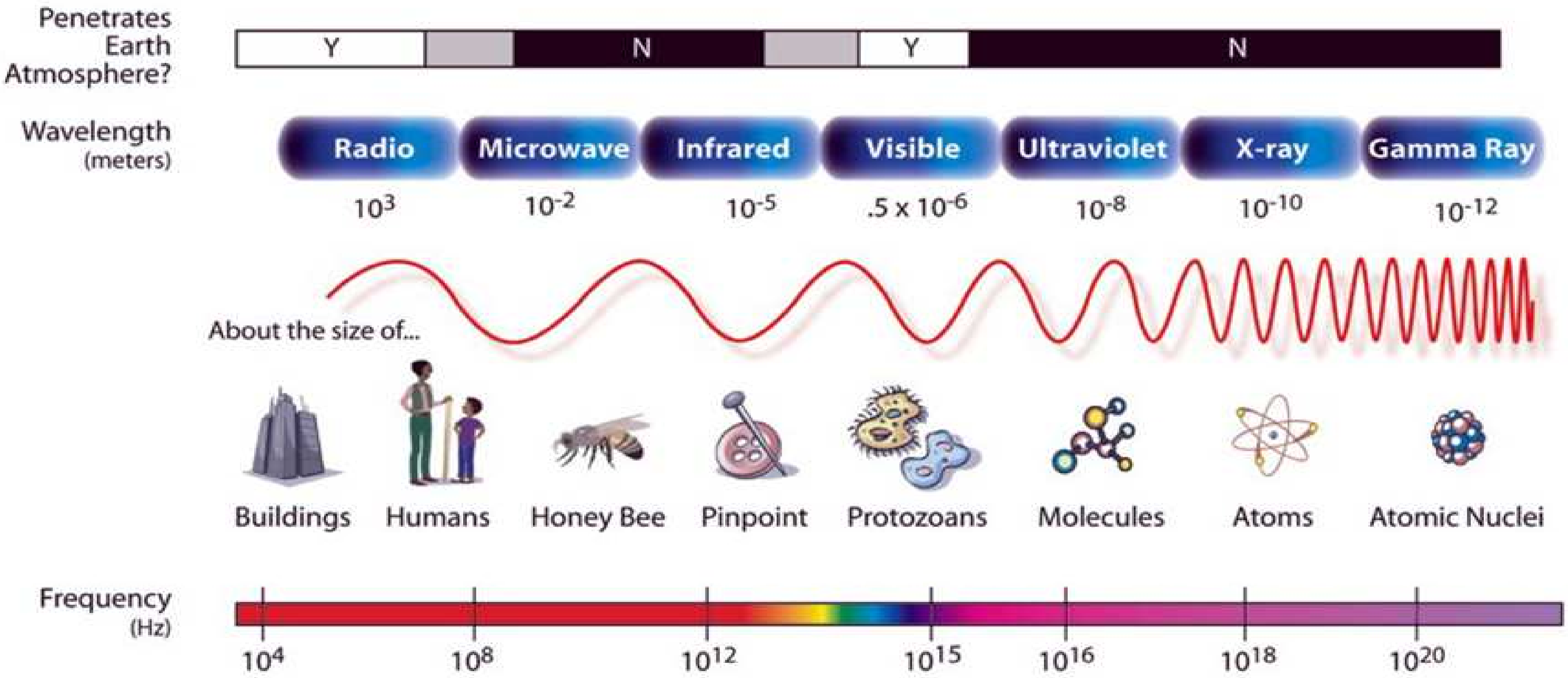
Frequency of visible light: 4.3×10^{14} **to** 7.5×10^{14} ***Hz***

Difference between a sound wave and a light wave?

Applet

One comes out of the radio the other is received by the radio.

Electromagnetic Spectrum -2



Why a single wave phenomenon is called so many names?

EM waves of different frequencies/lengths originate from different sources **and** interact with matter essentially differently**.**

Polarization -1

is a property of electromagnetic waves that specifies the electric field direction. Why do we pick the electric field out of the two? Aren't they on an equal footing?

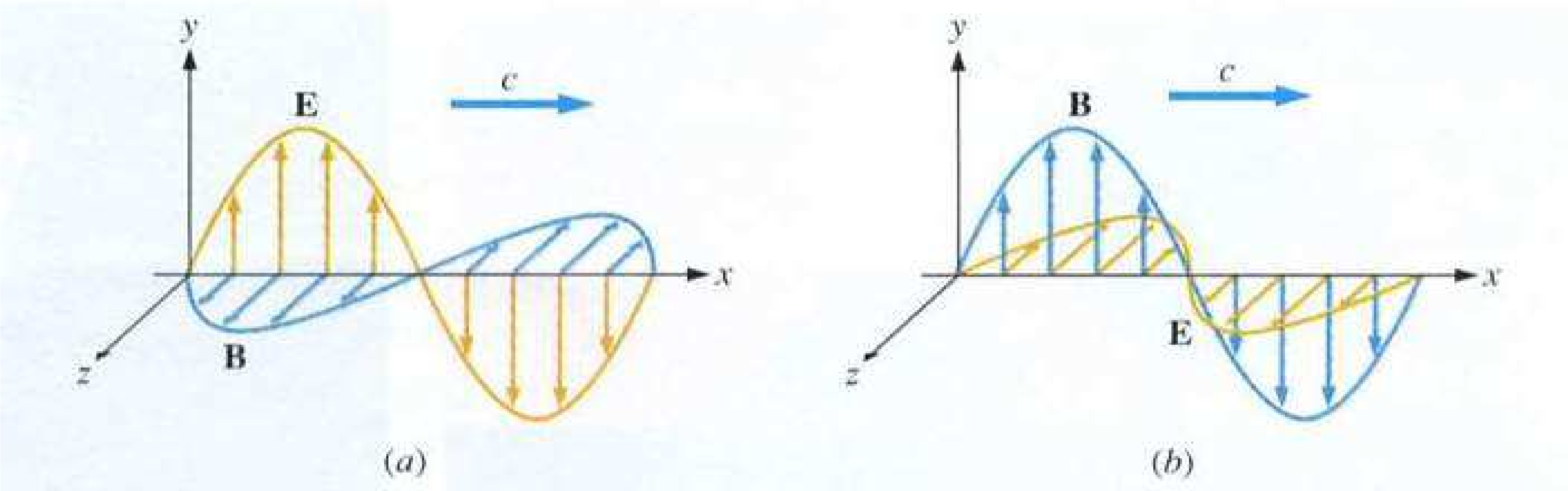


FIGURE 34-16 Field vectors for two electromagnetic waves. Both are propagating in the $+x$ direction, but have different polarizations. The polarization direction is that of the electric field, so the wave in (*a*) is vertically polarized (i.e., its electric field is in the y direction), while the wave in (*b*) is horizontally polarized (i.e., in the z direction). Unpolarized light would be a mix of such waves with their electric fields oriented at random in many different directions.

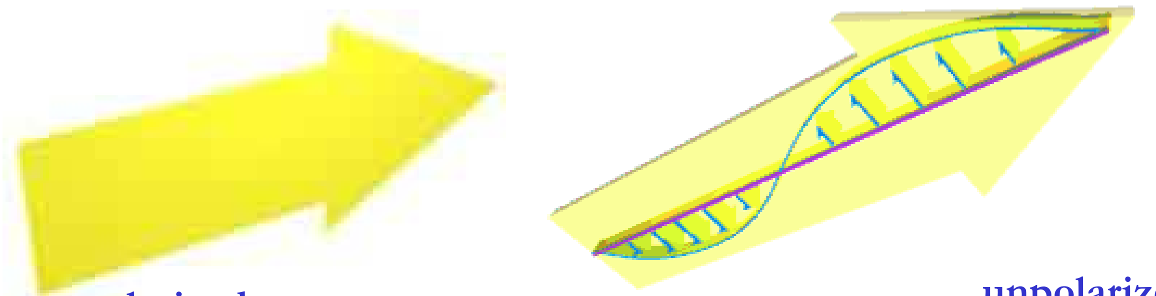
They are on an equal footing, but it is mostly the electric field, which interacts with matter – atoms, electrons, nuclei.

Polarization -2

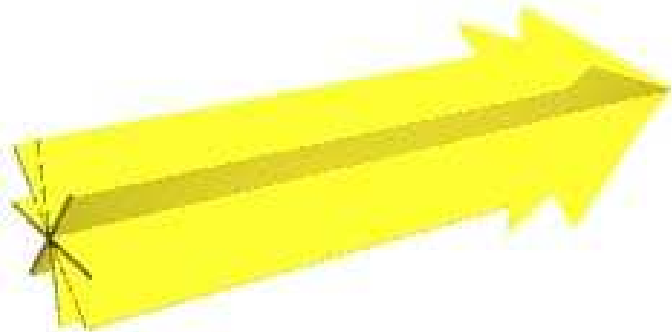
EM waves from a TV tower **are perfectly** polarized **– the Electric field has a very well defined direction, which stays always the same.**

In contrast, the light coming from the Sun or from a light bulb **is** unpolarized**. What does it mean unpolarized? Doesn't the electric field have some direction?**

It certainly does at every instant. BUT this **direction does not stay constant** and changes very rapidly and randomly.
So, averaging over any reasonable time interval you do not find any particular polarization!



polarized



unpolarized

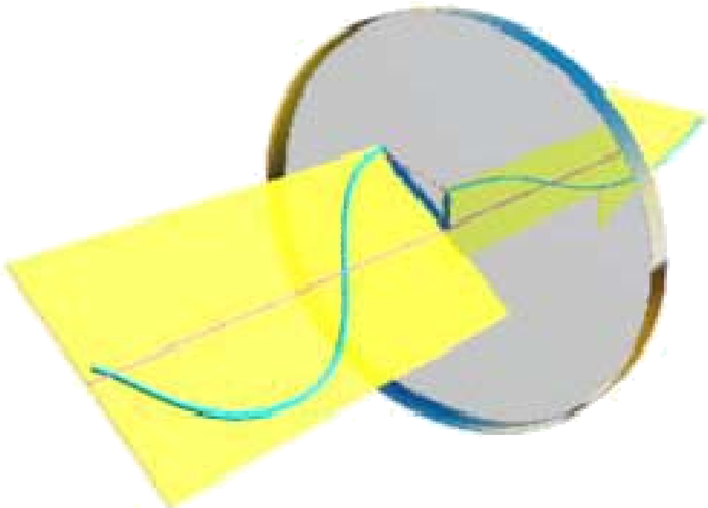
The frequency of light is about 5×10^{14} Hz, which means 5×10^{14} wave crests per second. If the polarization changes once every 500 crests it will still be 10^{12} times per second. Too fast for us to detect!

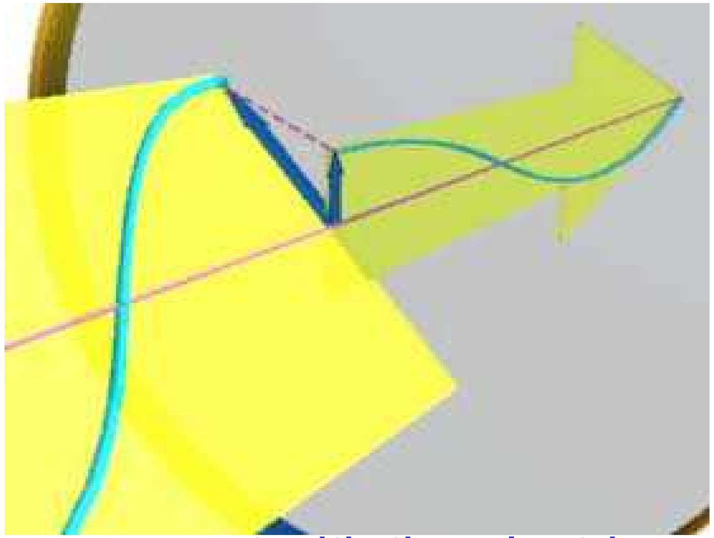
Polarizing the Light -1

Any way to make a polarized wave (light) out of unpolarized wave?

Yes, but it is going to cost us some intensity loss… (No free meals…)

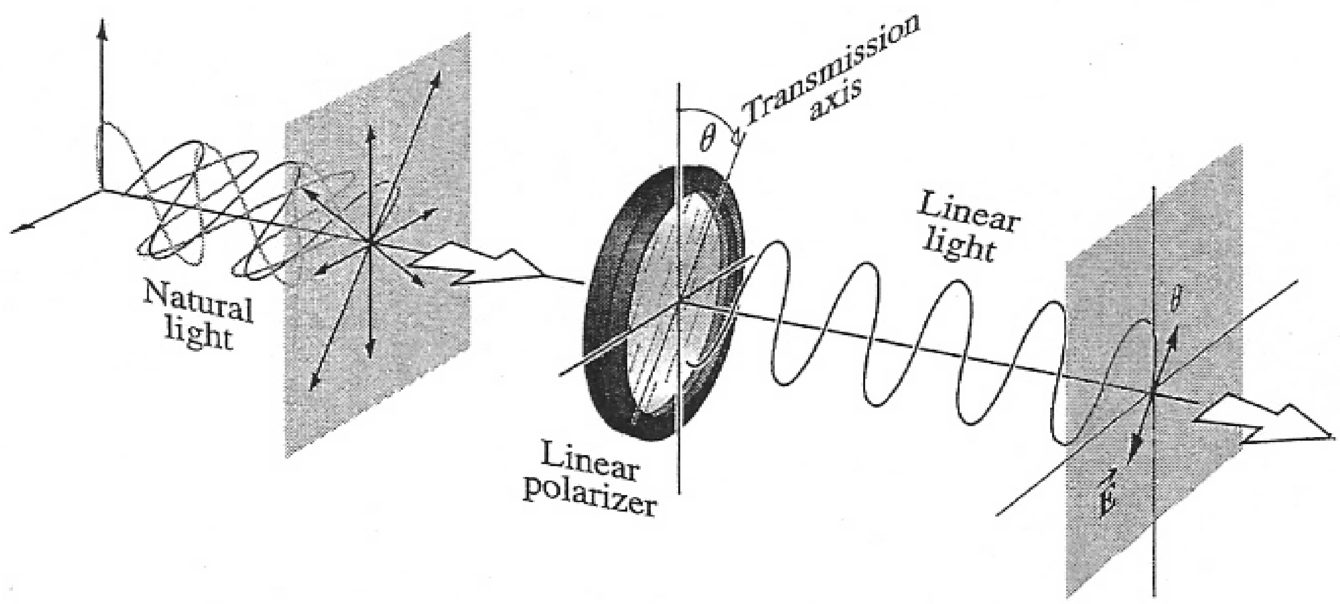
We can use a **polarizer** - a piece of material, whose molecular or crystal structure has a preferred direction called the **transmission axis.**





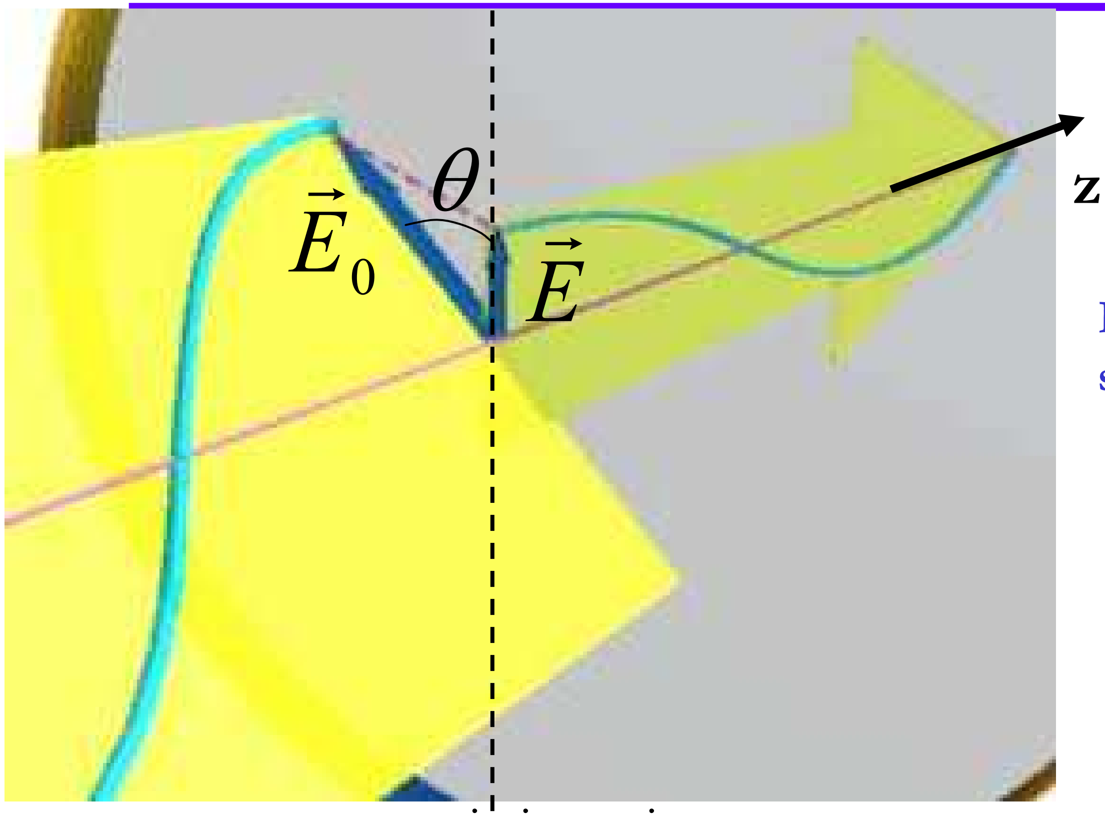
A polarizer "decomposes" the wave into a component with the electric field, $\mathbf{E}\|$, parallel to the transmission axis, which passes through, and a component with the $\mathbf{E}^{\perp}$ field perpendicular to the transmission axis, which gets totally absorbed.

Polarizing the Light - diff. view



Natural light incident on a **linear polarizer**: the transmitted light is only the **light component polarized** in the **plane** of **transmission axis**

Polarizing the Light -2



transmission axis

The magnitude of the right component of the electric field:

$$E = E_0 \cos\theta$$

Intensity of the wave is proportional to the square of the amplitude

$$S \sim E^2$$

$$S = S_0 E^2 / E_0^2 = S_0 \cos^2\theta$$

Law of Malus

In an unpolarized wave the angle θ is changing randomly. Therefore, after passing through a polarizer the average intensity is

$$\langle S \rangle = S_0 \langle \cos^2\theta \rangle = S_0 / 2$$

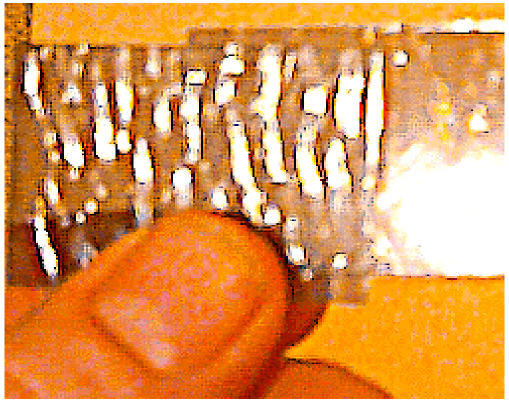
The light gets polarized, but it looses 1/2 of its intensity...

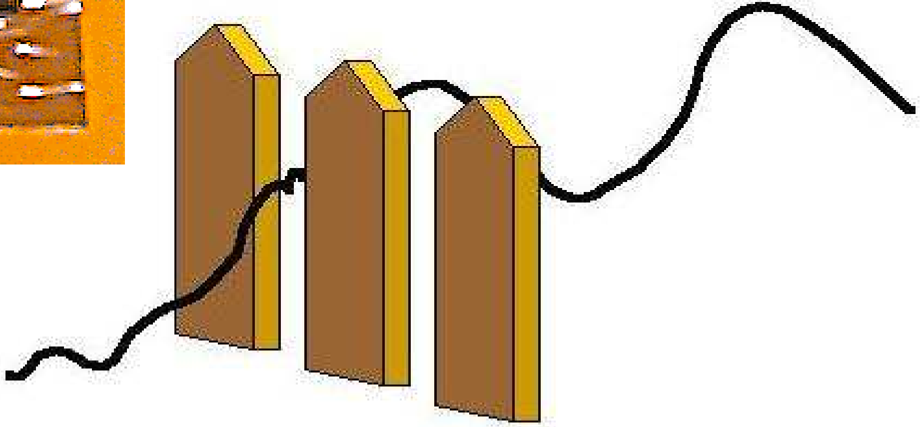
Crossing the polarizers

If the axis of a polarizer is set at $\theta = 90°$ to the axis of polarization:

$$\cos\theta = 0$$ **no light is passing through!**

A system of two crossed polarizers never lets any light through.
Whatever passes through the first one is blocked by the second.





Stack of polarizers

What happens to the intensity, S, and direction of polarization of unpolarized light upon passing trough three polarizers shown here?

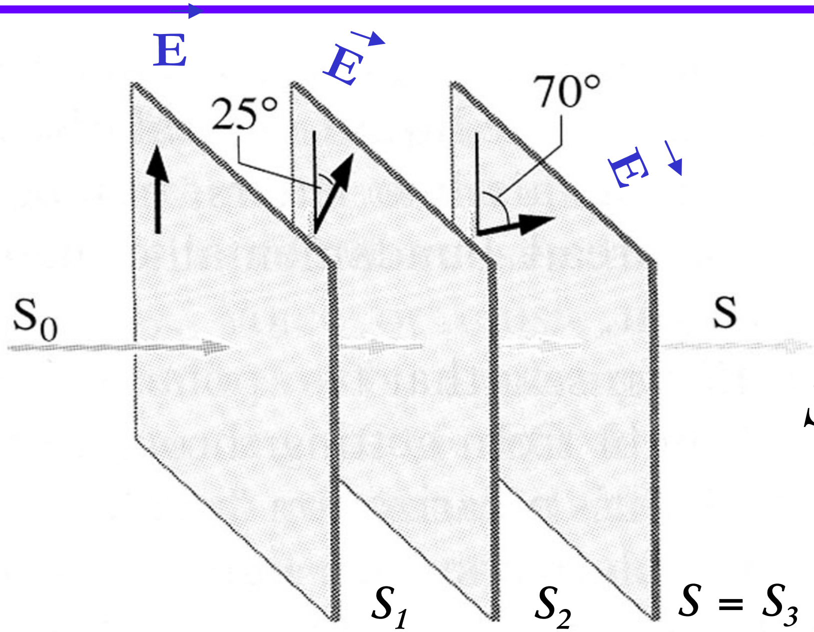


FIGURE 34-18 A stack of polarizers. Arrows on the sheets indicate directions of the polarization axes.

$$S_1 = S_0 / 2$$

$$S_2 = S_1 \cos^2 25°$$

$$S_3 = S_2 \cos^2 (70° - 25°)$$

$$= S_2 \cos^2 45°$$

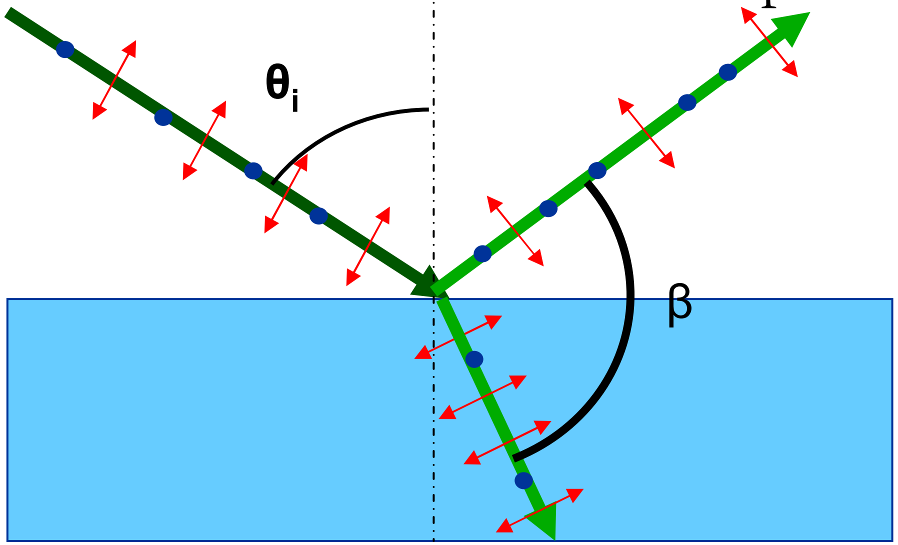
$$S_3 = S_0 \cdot 1/2 \cdot \cos^2 25° \cdot \cos^2 45° = 0.205 \cdot S_0$$

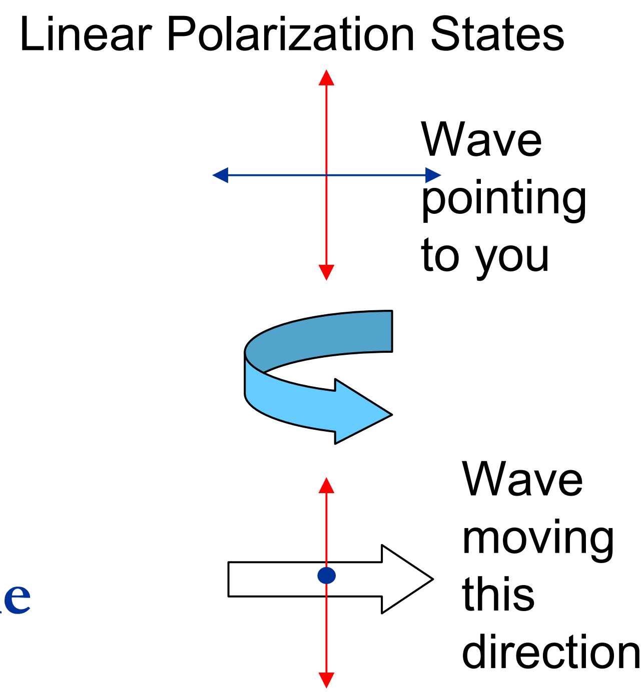
Without the second polarizer:

$$S_3 = S_0 \cdot 1/2 \cdot \cos^2 70° = 0.058 \cdot S_0$$

Polarization by Reflection from Dielectric Media

It's the most common source of polarized light





The reflected ray is **partially polarized**, in a **plane perpendicular** to the **incidence plane**.

For the incidence angle $\theta_i = \boldsymbol{\theta_B}$ **(Brewster's angle)** such that the angle between the reflected and refracted rays, $\boldsymbol{\beta}$, is $\mathbf{90^0}$, the reflected ray is **totally polarized !**

$$\boxed{\tan\theta_B = \frac{n_2}{n_1}}$$

Brewster's Angle Exercises

N1: Derive the Brewster's angle formula, $\tan(\theta_B)=n2/n1$, from the reflection and refraction laws.

N2: Which is the Brewster's angle for the air - water ($n=1.33$) surface ?